Eur. Phys. J. D **41**, 49–54 (2007)
DOI: 10.1140/epjd/e2006-00188-9

THE EUROPEAN PHYSICAL JOURNAL D

Interference and dispersion effects on tunneling time

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Received 24 November 2005 / Received in final form 21 July 2006
Published online 18 August 2006 –

Abstract. We study the interplay between pulse width, interference and tunneling for a wave packet incident upon a barrier and, within the context of tunneling time, we offer a complementary insight into the origin of the Hartman effect. We find that interference together with momentum spread lower (increase) the transmission (reflection) tunneling time thereby 'breaking the symmetry between transmission and reflection times'. But, within the limits of our method, we are unable to confirm that negative tunneling time can be obtained.

PACS. 03.65.Xp Tunneling, traversal time, quantum Zeno dynamics – 42.25.Bs Wave propagation, transmission and absorption – 73.40.Gk Tunneling

1 Introduction

In 1992 Martin and Landauer proposed to explore the analogy between the quantum tunneling of particles and evanescent electromagnetic waves in the study of particle tunneling [1]. This was motivated partly by the fact that, while the phenomena of interference and dispersion have been intensely studied within the context of wave propagation, their effects in particle tunneling had not received as much attention. This lack of attention is surely due to the dominant role played by the tunneling mechanism in attenuating the intensity of the beam as it tunnels through the barrier. But it would also have to do with the absence of consensus over tunneling time for which an actual measurement seems to be the only arbiter [2]. A related concern is that theoretical studies have focused on plane particle beams which, though simple, may not be easily available in an experiment [2,3]. Pulses, while more readily available in the laboratory, are less simple and might give rise to behavior that could only further confuse the unease over tunneling time. In this regard, we cite de Aquino et al. who showed that, as a result of deformations of the reflected and transmitted momentum distributions of a Gaussian wave packet incident upon a barrier, the reflection and transmission times of such packets display an unexpected dependence on their initial location [4]. Moreover, pulses naturally undergo dispersion so it is important to consider how barrier tunneling times might be affected by them. As we see below, dispersion and interference 'break the symmetry' between transmission and reflection tunneling times.

Perhaps inspired by Martin and Landauer and perhaps also due to the paucity of actual experimental results, workers have tended to invoke the analogy between frustrated internal reflection of electromagnetic waves and barrier penetration to gain insight into tunneling time [5]. Indeed, appealing to this analogy, one might consider how recent studies exhibiting unusual properties (i.e., negative transit time) in light-pulse propagation might find expression in tunneling phenomena [6]. As tempting as the analogy appears, we must note that the Schrödinger equation differs from the Helmholtz wave equation in that the former contains a derivative that is linear in time with an imaginary coefficient compared with the latter which has a second order time derivative with real coefficient: there is no a priori reason why the analogy can work well. In the following we will study how pulse width (i.e., momentum spread), dispersion and tunneling phenomena mutually interplay for a wave packet incident upon a barrier and, within the context of these phenomena, we offer a new complementary insight into the origin of the Hartman effect, that is, the saturation of the tunneling time as a function of barrier depth, a phenomenon occurring in both frustrated internal reflection and barrier tunneling [7–9]. (In optics the Hartman effect is studied under the aspect of the Goss-Hanchen effect [10].) At this time, the role of pulse width is hardly touched upon in connection with particle tunneling and, while we do not pretend to have the final word on the Hartman effect, the explanation we offer places the emphasis on barrier penetration together with interference instead of the more often discussed issue of wave propagation. We feel that the tendency to have the latter in mind when examining the Hartman effect is

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a source of some difficulty. However, we note that, within the limits of our method, we cannot confirm the reality of negative tunneling time.

The literature abounds with expressions for the tunneling time [2,3]. Our intention is *not* to compare them or favor one or another, but rather to exemplify the interdependence between tunneling, pulse width and dispersion. Thus we will employ those expressions for tunneling time that most efficiently bear on our purpose. With this in mind, in the next section we will make use of the phase time, whereas in Section 3 we will find it more convenient to employ Buttiker's expression for dwell time [3]. Section 4 discusses insights into the simultaneous interplay of interference and tunneling effects, which is the main focus of this report.

2 Tunneling time for a Gaussian pulse

We begin by recalling the transmission coefficient for plane waves of unit amplitude, and wave number k incident upon a barrier of height V_0 between $0 \le x \le a$ and zero elsewhere [11],

$$T(k,\kappa,a) = -e^{-\kappa a}\frac{2i\sin 2\theta}{D(k,\kappa,a)}e^{-ika}e^{-i\alpha}$$
$$D(k,\kappa,a) = \sqrt{1+e^{-4\kappa a}-2e^{-2\kappa a}\cos 4\theta} \quad (1)$$

in which m is the particle mass, $k^2 = 2mE/\hbar^2$, $k_0^2 = 2mV_0/\hbar^2$, $\kappa^2 = k_0^2 - k^2 \ge 0$, $\cos\theta = k/k_0$ and the tunneling phase angle is $\alpha = \tan^{-1}(\coth\kappa a\tan 2\theta)$. Next we assume an incident Gaussian-like wave packet whose center at $x = -l$ is far from the barrier, i.e., $\left(2\sigma^2/\pi\right)^{1/4} e^{i\tilde{k}(x+l)}e^{-\sigma^2(x+l)^2/4}$ which corresponds to a momentum distribution centered at $\tilde{k}$:

$$g(k-\tilde{k})e^{ikl} = \frac{1}{\left(2\pi\sigma^2\right)^{1/4}}e^{ikl}e^{-(k-\tilde{k})^2/4\sigma^2}. \quad (2)$$

Then the wave transmitted $\Psi_T(x,t)$ beyond the barrier at $x > a$ is given by

$$\Psi_T(x,t) = \frac{1}{\sqrt{2\pi}}\int_0^{k_0} g(k-\tilde{k})e^{-\kappa a}\frac{-2i\sin 2\theta}{D(k,\kappa,a)} \times e^{ik(x+l-a)}e^{-i\omega t}e^{-i\alpha}dk \quad (3)$$

where $\omega(k) = E/\hbar = \hbar k^2/2m$. Like de Aquino at al. [4], we consider only momenta corresponding to tunneling into the barrier and like them we will assume that extending the range of k to $-\infty$ and $+\infty$ in equation (3) will alter the result insignificantly. This requires the distribution width σ to be small enough.

The above integral will be computed approximately using the method of steepest descent instead of the more often used stationary phase approximation [12]. Let us first expand the exponents about $k = \tilde{k}$ and write the phase in the form $\alpha = \tilde{\alpha} + \dot{\tilde{\alpha}}(k-\tilde{k}) + O(k-\tilde{k})^2 = \alpha_0 + \dot{\tilde{\alpha}}k + O(k-\tilde{k})^2$. Completing the square, we recast the integral in the form

$$\Psi_T(x,t) \cong \frac{1}{\sqrt{2^{3/2}\pi^{3/2}}\sigma}e^{-i(\bar{E}t+\alpha_0)}e^{i\bar{k}(x+l-a-\dot{\tilde{\alpha}})}$$
$$\times\exp\left\{-\frac{1}{2}\frac{\left(x+l-a-\frac{\hbar\bar{k}}{m}t-\dot{\tilde{\alpha}}\right)^2}{\frac{1}{2\sigma^2}+i\frac{\hbar t}{m}}\right\}$$
$$\times\int_0^{k_0}\exp\left\{-\frac{1}{2}\left(\frac{1}{2\sigma^2}+i\frac{\hbar t}{m}\right)\left(k-\bar{k}+i\frac{x+l-a-\frac{\hbar\bar{k}}{m}t-\dot{\tilde{\alpha}}}{\frac{1}{2\sigma^2}+i\frac{\hbar t}{m}}\right)^2\right\}$$
$$\times\, t(k,\kappa,a)e^{-i\pi/2}dk \quad (4)$$

in which the barred quantities are computed at $k = \tilde{k}$. We go a step further than de Aquino [4] by including the tunneling factor, i.e. expanding κ to second order about $\tilde{k}$. If we also expand the tunneling factor $e^{-\kappa a}$ about $k = \tilde{k}$, we obtain exactly the same result as (4) except that $\tilde{k}$ and $1/2\sigma^2$ are now replaced by

$$\tilde{k} \to \bar{k} \equiv \tilde{k}\left(1+\frac{2a\Sigma^2}{\tilde{\kappa}}\right)$$

and

$$\frac{1}{2\sigma^2} \to \frac{1}{2\Sigma^2} \equiv \frac{1}{2\sigma^2} - \frac{k_0^2 a}{\tilde{\kappa}^3}, \quad (5)$$

respectively and

$$t(k,\kappa,a) \cong -2ie^{\left(\Sigma\bar{k}a/\tilde{\kappa}\right)^2}e^{-\tilde{\kappa}a}\sin 2\bar{\theta}/D(\bar{k},\tilde{\kappa},a).$$

(Note: by definition Σ^2 can be negative.) Thus, in equation (4) we will now understand that the barred quantities are to be evaluated at $k = \bar{k}$. Observe, however, that $\tilde{\alpha}$ and $\tilde{\kappa}$ are computed at $k = \tilde{k}$.

Since the quantity $1/2\sigma^2$ is assumed to be large, the asymptotic behavior of the integral may be found by the method steepest descent. When this is applied to equation (4), we realize that it would affect neither the phase relations nor the Gaussian envelope at the front of the integral. Moreover, it is from these quantities that the information about the tunneling time is to be obtained. Following de Aquino [4], we obtain the tunneling time due to transmission,

$$\tau^T = -\frac{m}{\hbar\bar{k}}\left.\frac{d\alpha}{dk}\right|_{k=\tilde{k}}. \quad (6)$$

We see that, as a result of tunneling, the mean wave number of the transmitted wave has been shifted from $\tilde{k}$ to a higher value $\bar{k}$; that is, from equation (4), the shift $\bar{k}-\tilde{k}$ is positive, linear in the width of the barrier and quadratic in the momentum spread Σ. This latter behavior had been noted in numerical studies by de Aquino et al. [4]. The shift is expected because the higher-k components of the incident packet have a smaller barrier height to tunnel through than the lower-k components. Note finally that except for the change from σ to Σ, the effect of dispersion, as contained in the quantity $1/2\Sigma^2 + i\hbar t/m$, is the expected one [13].

A similar calculation can be carried out for the reflected wave. The reflection coefficient for a rectangular barrier is [11]

$$R(k,\kappa,a) = \frac{(e^{-2\kappa a}-1)}{D(k,\kappa,a)} e^{-i\alpha}, \quad (7)$$

so the reflected wave packet $\Psi_R(x,t)$ corresponding to equation (2) is

$$\Psi_R(x,t) = \frac{1}{\sqrt{2\pi}} \int_0^{k_0} g(k-\tilde{k}) \frac{(e^{-2\kappa a}-1)}{D(k,\kappa,a)} \times e^{-ik(x-l)} e^{-i\omega t} e^{-i\alpha} dk. \quad (8)$$

Provided $\kappa a > 1$ (this will suffice for our discussion) we may carry out a steepest descent calculation readily by replacing $R(k,\kappa,a)$ by $-e^{-i\alpha}\exp[-2e^{-2\kappa a}\sin^2 2\theta]$. (This last approximate expression is found by expressing the numerator and denominator of Eq. (7) as exponentials.) The result is just equation (4) with $x-a$ replaced by $-x$ in the exponents and the factor $e^{-i\pi/2}t(k,\kappa,a)$ is replaced by -1. In place of equation (5), the barred quantities are now evaluated at

$$\bar{k}' \equiv \tilde{k}\left(1 - \frac{2a\sigma^2}{\tilde{\kappa}}(2\sin 2\theta)^2 e^{-2\kappa a}\right), \quad (9)$$

the change in σ being negligible. The reflection time τ^R is just equation (6) with $\bar{k}$ replaced by $\bar{k}'$. The shift in momentum of the reflected wave $\bar{k}' - \tilde{k}$ is now negative, but as with the transmission shift, it is also linear in barrier width and quadratic in the momentum spread. Both momentum shifts grow with increasing incident momenta, unlike the shifts calculated by de Aquino et al. [4]. However, their results apply to the distribution in momentum space whereas ours corresponds to the mean momentum of the Gaussian envelope in position space. In an actual experiment it is this latter situation that is directly accessible.

Graphs of the transmission and reflection times for various values of σ are shown in Figures 1 and 2 for a particle whose kinetic energy is half the barrier height. Evidence for the Hartman effect is exhibited by the flat portions of the $\sigma = 0$ graphs for larger barrier widths a. The saturation of the tunneling time displayed in Figure 1 applies only for (approximately) monochromatic wave packets, with zero momentum spread. This implies that the wave packet is able to sample the front and back ends of the barrier almost simultaneously, since the spatial extent of the barrier is much smaller than that of the wave packet's. As momentum spread σ increases, the transmission (reflection) time dips (rises) with increasing widths. The effect is less noticeable for reflection time for reasons we will discuss in Section 4. Hence we see a suggestion that τ^T and τ^R are not equal. This supposed equality holds for a plane wave, but it can longer be true for a pulse. Notice also that the tunneling times are approximately linear in barrier width up to a depth a given by $\kappa a \approx 1$. These observations warrant a study into the Hartman effect, which we embark on in the next section.

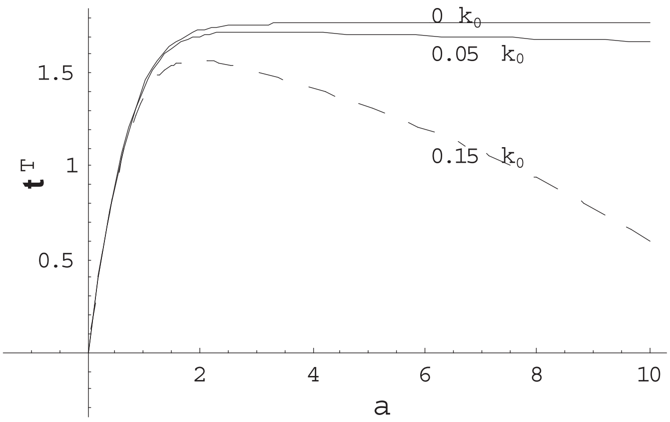

Fig. 1. Barrier transmission times τ^T (in arbitrary units) versus barrier width a (in units of $1.6/k_0$) for an incident particle with kinetic energy equal to half the barrier height. The momentum spreads are $\sigma = 0, 0.05k_0, 0.15k_0$ respectively. The flat portion of the top curve is a manifestation of the Hartman effect and the linear behavior for small a is consistent with equation (16). The penetration depth is $d \approx \kappa^{-1} \approx 2/k_0$ (see discussion preceding Eq. (18)). The inference of negative tunneling time is inconsistent with our calculation (see end of Sect. 4).

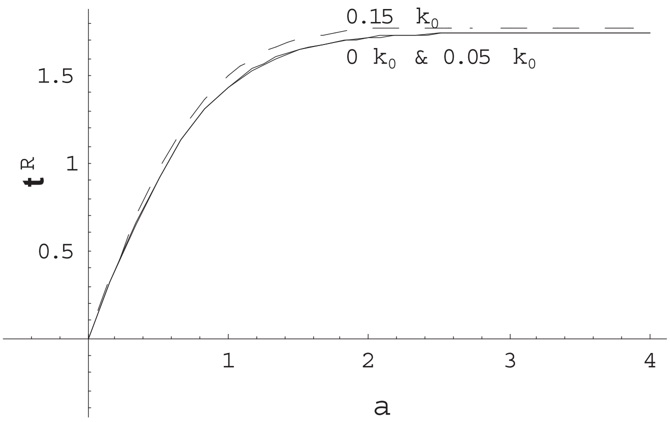

Fig. 2. Same as Figure 1 but for barrier reflection times τ^R (in arbitrary units) versus barrier width a (in units of $1.6/k_0$).

3 Hartman effect

In Young's classic experiment, light from a coherent source propagates through two slits that split the light into two component beams which in turn traverse separate paths and interfere at a screen. The path difference between the beams gives rise to a phase difference that is manifested in the interference fringes observed on the screen. This gives a measure of the path difference and hence the separation between slits. To determine the tunneling time through a barrier, we putatively identify the two ends of the barrier with the slits and study the interference produced by the wave entering the barrier with the wave reflected from the exit end. Thus a tunneling time associated with the full barrier length requires interference from the two ends. However, for interference to be discernible, the intensities of the beams from these ends must be roughly the same at the interference point. (In reality, of course, tunneling, being an intrinsically time dependent process, must be treated in a time-dependent way in which reflection and propagation take place all along the length of the barrier and not only at the ends.) In the case of an infinitely wide barrier there is no reflection from 'infinity' and, hence, no interference between the 'ends'; for a finite but wide barrier, the reflected intensity at the exit end is weak so its interference with the entering beam is hardly distinguishable from the former case. This is the operational reason for the Hartman effect: the interference produced in a wide barrier is hardly different from that of an infinitely wide barrier implying almost identical path differences between the incident and reflected waves and hence (*if* this interference is associated with tunneling time) unlimited tunneling speed. Among other things, this will assist us in understanding why the transmission and reflection tunneling times are not identical as is generally thought. This can be seen from Figures 1 and 2: the transmission and reflection tunneling times are no longer equal in the dispersive case ($\sigma \neq 0$). See also reference [14]. We now quantify these remarks and apply them to the discussion at the end of Section 2.

For plane waves, the reflection and transmission coefficients R, T given by equations (1) and (7) can be cast in the form,

$$R = R_0 \left(1 + \frac{(R_0^2 - 1)e^{-2\kappa a}}{1 - R_0^2 e^{-2\kappa a}}\right)$$

$$T = (1 - R_0^2)\frac{e^{-ika}e^{-\kappa a}}{1 - R_0^2 e^{-2\kappa a}} \tag{10}$$

where $R_0 = (k - i\kappa)/(k + i\kappa)$. It is immediately evident that R_0 is the reflection coefficient when the barrier is infinitely thick, i.e. $a \to \infty$. We also introduce the transmission coefficient for a particle incident on such an *infinitely* thick barrier,

$$T_0 = \frac{2k}{k + i\kappa} \tag{11}$$

as well as analogous coefficients R_0', T_0' when the particle is incident on the barrier wall from *inside* the barrier,

$$R_0' = -R_0, \quad T_0' = i\frac{\kappa}{k}T_0. \tag{12}$$

One verifies that R and T can be cast as geometric series in terms of the barrier-side reflection and transmission coefficients R_0, T_0, R_0', T_0' and the tunneling factor $e^{-\kappa a}$,

$$R = R_0 + T_0 e^{-\kappa a} R_0' e^{-\kappa a}\left(1 + R_0' e^{-\kappa a} R_0' e^{-\kappa a} + ...\right) T_0' \tag{13}$$

$$T = e^{-ika} T_0 e^{-\kappa a}\left(1 + R_0' e^{-\kappa a} R_0' e^{-\kappa a} + ...\right) T_0'. \tag{14}$$

We interpret equation (13) as follows: the first term, R_0, is due to reflection from the entrance of the barrier; the next term, $T_0 e^{-\kappa a} R_0' e^{-\kappa a} T_0'$, contains a transmission factor at the entrance T_0, a tunneling factor $e^{-\kappa a}$ as the particle tunnels toward the other end, a reflection factor R_0' at this end, another tunneling factor $e^{-\kappa a}$ as the particle tunnels back toward the entrance, and finally a transmission factor T_0' for the particle to exit the barrier; etc. A similar interpretation can be given for equation (14). The above series are similar to those encountered in multiple-wave interference in physical optics. Equations (13) and (14) are not new but our quantitative interpretation of the tunneling time in terms of them is new.

To proceed, it will be more convenient to use Buttiker's characteristic time τ_y instead of the phase time [3],

$$\tau_y^T = -\frac{m}{\hbar\kappa}\frac{\partial}{\partial\kappa}\alpha. \tag{15}$$

In pursuit of the Hartman effect, two limits are interesting to study in some detail. When the barrier is *thin* we find from equation (15)

$$\tau_y^T \to \frac{m}{\hbar k}a = \frac{a}{v} \quad \text{(thin)}. \tag{16}$$

where v is the classical velocity. This means that Buttiker's dwell time is just the time for the particle to move classically through the barrier, as if the potential did not exist. The tunneling time is thus sensible and meaningful. Following Winful [15], we can also compute the self-interference delay defined by$\tau_i = -\hbar \mathrm{Im}(R)(d\ln k/dE)$, which is a consequence of the overlap of the incident and reflected waves at the front of the barrier (this expression applies to both massive particles and photons but is strictly valid for symmetric structures [16]). We obtain

$$\tau_i \to \frac{a}{v}\left(\frac{V_0}{2E}\right) \quad \text{(thin)} \tag{17}$$

which is roughly equal to τ_y^T, so that the (thin-barrier) tunneling time (16) is largely accounted for by the wave interference at the barrier entrance.

The other limit is when the barrier is *very wide*. In this case the first terms of equations (13) and (14) are sufficiently good approximations of R and T. Therefore the characteristic times computed for such a barrier are practically the same as those for an infinitely wide one because the interference effects in T and R are effectively nullified when the tunneling factor $e^{-\kappa a}$ is small. Define a barrier penetration depth $d = \kappa^{-1} \ll a$. Then

$$\tau_y^T \to \frac{E}{V_0}\frac{2d}{v} \quad \text{(wide)}, \tag{18}$$

and the self-interference delay is

$$\tau_i \to \hbar \mathrm{Im}(R_0)\frac{d\ln k}{dE} = \left(1 - \frac{E}{V_0}\right)\frac{2d}{v} \quad \text{(wide)}. \tag{19}$$

The characteristic time (18) in this case refers to the wave penetrating into the barrier to a depth of order d whereas the calculation of (19) refers to a single reflection event at the front of the barrier. Neither time probes the entire width of the (thick) barrier. This is unlike the case for thin barriers. At the end of Section 2, we had pointed out the linear behavior of $\tau^{T,R}$ versus a (for small a) in Figures 1 and 2; we see that this is consistent with the criterion given for the penetration depth as well as equation (16).

For thick barriers, the Hartman effect is essentially due to the suppression of all but the first terms of equations (13) and (14) because of the smallness of the tunneling factor: the interference between waves from the front and back ends of the barrier is largely negligible; equivalently, the wave penetrates significantly only to a depth of order d into the barrier. That is, for the transmission time τ_y^T and for an analogous reflection time τ_y^R, the relevant interference arises from waves within a depth of order d from the front end of the barrier. Then, for wide barriers ($a \gg d$), the characteristic time is independent of width. Clearly in this case the characteristic time cannot be regarded as the propagation time through the barrier as noted already.

What is the operational meaning of the penetration depth? It is the depth to which waves interfere effectively so as to create the dominant reflected wave and hence the dominant phase required to determine time. Recall from the beginning of this section that tunneling time required effective interference between the 'ends of the barrier.' This gives rise to a phase and a time delay. For a thick barrier these 'ends' are separated by the penetration depth. Application of the formulas of this section to frustrated internal reflection are discussed in reference [17].

4 Discussion

In Figure 1, we had noted the decrease in transmission tunneling time with increasing barrier width a and increasing momentum spread σ. This is due to the shift in mean momentum $\bar{k} - \tilde{k}$ on account of tunneling and pulse width (cf. Eq. (5)). The dependence of the shift on barrier depth, a strictly tunneling phenomenon, stems from the fact that as the barrier width is increased, fewer of the slower-k components are able to tunnel through the barrier, leaving it to more of the higher-k components to go through. The dependence of shift on momentum spread is the more dominant (appearing quadratically) because (a) greater σ generates a higher mean momentum on account of the tunneling feature just mentioned, and (b) a higher mean momentum implies also a shorter pulse tunneling time. Similar remarks can be made about the reflection time, but in the opposite direction, i.e. an increase in time (cf. Eq. (9)).

As to why the effect is noticeably smaller for the reflection time, recall that the derivation of equation (9) required us to take both terms in the numerator of the reflection coefficient $R(k, \kappa, a)$ as well as the two dominant terms in the denominator (cf. Eq. (7)). That is, we had to account for the interference between waves reflected from the interior of the barrier *and* waves reflected from the front end of the barrier. (It is easier to see this in Eq. (10) where R_0 is the reflection coefficient from the front end.) Hence, together with tunneling we also have interference to take into account. This was not necessary to include in the transmission case (indeed Eq. (10) shows that T is entirely due to internal reflections inside the barrier). This difference between the ways T and R are treated is due to interference and explains the difference between the graphs of Figures 1 and 2 and hence the "loss of symmetry between τ^T and τ^R". We discuss this interplay of tunneling and interference in some detail in the following paragraph. Our task will be to explain why equation (9) contains a tunneling factor $e^{-2\kappa a}$ which equation (5) does not. Note that although R and T refer to a plane wave, by virtue of equations (3) and (8), their effects apply just as well to pulses. The equality of τ^T and τ^R is usually thought to be due to the unitary of the scattering matrix [18].

It could be argued that there is really no symmetry between the transmission and reflection times because the transmission process is independent of directionality whereas reflection is sensitive to left-to-right and right-to-left geometry [19]. Although this is a powerful reason, we have gone further here by suggesting that this lack of symmetry is connected with pulse width (cf. Figs. 1 and 2) and interference (see below).

We had argued at the end of Section 3 that the Hartman effect is due to a failure in the interference between waves at the front of the barrier and those reflected from the*end* of a wide barrier. To be precise, the effective interference takes place mainly between the waves at the front of the barrier and those reflected from within a depth κ^{-1} of the front (cf. end of Sect. 3). We saw also in the previous paragraph that on account of the tunneling of a pulse, it is the slower-k components that will populate the reflected wave. But smaller k implies greater κ and hence shallower effective depth of the interference region. Therefore the effect of increasing momentum spread is to reduce the depth of effective interference. As the barrier widens, less of the incident wave is able to tunnel through and consequently more interference must take place at the front region; this is exemplified by the numerator of the reflection coefficient (Eq. (7)). But this cannot go on unabated because for an infinitely wide barrier, all the waves are reflected back. This then accounts for the presence of the tunneling factor $e^{-2\kappa a}$ in equation (9): the region beyond the penetration depth κ^{-1} from the front has an exponentially decaying contribution to the overall interference occurring at the front end of the barrier.

A final word about negative tunneling time. For this to occur, equation (5) implies that the quantity $a\Sigma^2/\tilde{\kappa}$ should be large and negative. In fact, $a\Sigma^2/\tilde{\kappa}$ becomes very large when $k_0 a \left(\sigma/k_0\right)^2 / \sin^3\theta \approx 1$. But this is inconsistent with the expansion of the tunneling factor $e^{-\kappa a}$ to second order, which requires this same quantity to be less than unity. Also we had seen in the previous paragraph that ultimately all the waves will be reflected back for an infinitely wide barrier, implying that the momentum shift cannot grow indefinitely. Hence it appears that our method is inapplicable in deciding this case with confidence.

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